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Accurate Prediction of Critical Conditions for Shear-Loaded Panels

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GENERAL instability of eccentrically stiffened thin cylindrical panels subjected to uniform axial compression, uniform pressure, and shear has been investigated by Simitses.[1]. The procedure employed is similar to those used in the study of constant-thickness isotropic cylindrical panels.[2-4]

In the case of uniaxial or biaxial compression with classical simply supported boundaries, the solution to the buckling equations is taken as $\sin(m\pi x/L)\,\sin(n\pi y/b)$ and the critical load is obtained through minimization with respect to the wave numbers m and n. In the case of shear or combination of shear with uniaxial or biaxial compression, because of the nature of the buckling equation, usually a Rayleigh-Ritz or a Galerkin procedure is employed. The transverse displacement is represented by a double Fourier series,[1,2] and the critical condition is obtained from the solution of a determinant (characteristic equation) which is dependent on the number of terms taken in the series representation of the transverse displacement; the larger the size of the determinant, the better will be the approximation. But, as the size of the determinant is increased, the computer time required to calculate critical conditions increases rapidly. It therefore is desirable to keep the size of the determinant as small as posssible without sacrificing accuracy. Schildcrout and Stein[2] report that 10 terms in the series representation usually are sufficient to predict critical conditions accurately. Of course they meant the first 10 terms, and this is questionable, especially in the case of stiffened configurations.

The requirement of keeping the size of the determinant small and thus the computer time required low is extremely important in the optimization of stiffened panels and individual or combined load conditions, which include shear. This is so because, in the process of arriving at the optimum configuration, it is necessary to compute critical conditions (solve the determinant) for numerous combinations of the geometric parameters. The present paper deals with the problem of finding a reasonable computational method (from the point of view of computation time) to compute critical conditions accurately for shear loaded panels. This is accomplished by keeping the size of the determinant low through the retention of the most influencing terms in the Fourier expansion for the transverse displacement. In addition, the effect of the different geometric parameters on the size and type of the needed determinant is investigated.

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The governing equations[1] are

$$A_{mn}\alpha_{mn} + k_s \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} \frac{A_{m'n'}\, mn\, m'n'}{[(m')^2 - m^2]\,[(n')^2 - n^2]} = 0 \tag{1}$$

for $m,n = 1,2$ where $m \pm m' = \text{odd}$, $n \pm n' = \text{odd}$, and where

$$\alpha_{mn} = (\pi^2/32\xi)\,[p(m,n) - k_x m^2 - k_y \xi^2 n^2] \tag{2}$$

where ξ is the panel aspect ratio (L/b), p is a function of m,n and the geometric parameters, and k_x, k_y, k_s are the load parameters.[5] Equation (1) can be decoupled into two systems, one corresponding to $m \pm n = \text{even}$ and one $m \pm n$ odd. The former is referred to as symmetric buckling and the latter as antisymmetric buckling. Both characteristic equations (determinants) must be solved in order to find the critical condition for a given set of geometric parameters.

In general, each one of the two determinants, one for symmetric buckling and one for antisymmetric bucking, can be written in the following matrix form:

$$\{[A] + k_s[B]\}\ \{X\} = 0 \tag{3}$$

where $\{X\}$ is the displacement vector that consists of the Fourier coefficients A_{mn}, $[A]$ is a diagonal matrix containing α_{mn} only, and $[B]$ is a matrix that includes all of the remaining terms and has zero elements along the diagonal. Since both matrices are symmetric and since matrix $[A]$ is positive definite (this is definitely true for shear loaded panels, but it is true also for the case of combined shear with other destabilizing loads, k_x and k_y, provided that the panel does not buckle when $k_s = 0$), then the eigenvalue k_s is always real.[7]

The eigenvalue k_s and the corresponding eigenvector $\{X\}$ can be calculated by matrix iteration. Since the sign of the eigenvalue does not affect the solution, it is calculated by employing a Rayleigh quotient[6] in the form

$$k^2_{s_{j+1}} = \{V_{2j+2}\}^T\{V_{2j}\}/\{V_{2j+2}\}^T\{V_{2j+2}\} \tag{4}$$

where $j = 0,1,2,\ldots$(iteration number), and $\{V_{j+1}\}$ is calculated from the equation

$$[A]\{V_{j+1}\} = -[B]\{V_j\} \tag{5}$$

by arbitrarily choosing the initial vector $\{V_0\}$. The corresponding eigenvectors are given by

$$\{X_1\}_{j+1} = \{V_{2j+1}\} + k_{s_{j+1}}\{V_{2j}\} \qquad \text{for positive } k_s \tag{6a}$$

$$\{X_2\}_{j+1} = \{V_{2j+1}\} - k_{s_{j+1}}\{V_{2j}\} \qquad \text{for negative } k_s \tag{6b}$$

This iteration procedure is continued until the change in the computed eigenvalue, δk_s, from two successive iterations, is smaller than a specified number ϵ (convergence criterion). Through this well-known procedure, the desired accuracy for the eigenvalues is achieved, but through the corresponding eigenvector one easily can assess its most influencing elements (A_{mn}).

It is conjectured that, if the the most influencing term in the eigenvector is $A_{k\ell}$, then 1) as the subscripts m and n move away from k and ℓ the relative magnitude of A_{mn} diminishes, and 2) if all of the A_{mn} terms with relative magnitudes smaller than 10% of the magnitude of $A_{k\ell}$ are excluded from the

Table 1 Critical shear stress for an unstiffened panel ($\xi = 0.187, Z = 2000, \nu = 0.33$)

Range of m and n (input)		Symmetric buckling			Antisymmetric buckling			Computer CPU time, mlsec.
		$k_{s_{cr}}$	Dominant term of the eigenvector $A_{k\ell}$		$k_{s_{cr}}$	Dominant term of the eigenvector $A_{k\ell}$		
m	n		k	ℓ		k	ℓ	
1-6	1-6	58,560.0	5	5	58,530.0	4	5	360
1-9	1-9	22,280.0	8	8	22,240.0	7	8	862
1-13	1-13	9,037.0	1	13	9,061.0	1	12	2233
1-4	1-19	1,912.0	1	19	1,912.0	1	18	641
1-5	1-30	367.0	1	29	364.5	1	28	2564
1-5	1-37	278.6[a]	2	36	277.2[a]	1	36	3300
1-3	1-50	272.9[a]	1	35	272.3[a]	1	36	2550
1-5	9-20	1,524.0	1	19	1,528.0	1	20	481
1-5	15-26	545.0	1	25	544.6	1	24	481
1-5	15-38	272.8[a]	1	37	273.2[a]	1	37	1663
1-5	25-48	266.0[a]	1	35	265.0[a]	1	36	1663
1-3	31-40	273.0[a]	1	37	276.1[a]	1	34	316
1-3	27-50	272.4[a]	1	35	272.7[a]	1	36	736
1-4	31-38	274.2[a]	1	35	275.1[a]	1	34	234

[a] Acceptable values for $k_{s_{cr}}$.

Table 2 Critical conditions for an unstiffened panel under combined shear and axial compression ($\xi = 0.667, Z = 1, \nu = 0.33$)

Range of m and m		k_x	Symmetric buckling			Antisymmetric buckling			Computer CPU time, mlsec.
			$k_{s_{cr}02}$	Dominant term of the eigenvector $A_{k\ell}$		$k_{s_{cr}}$	Dominant term of the eigenvector $A_{k\ell}$		
m	n			k	ℓ		k	ℓ	
1-5	1-5	1.8	2.956	1	1	6.186	1	2	213
1-5	1-5	-1.8	9.572	1	1	9.614	1	2	233
1-3	1-3	1.8	2.959	1	1	6.189	1	2	110
1-3	1-3	-1.8	9.577	1	1	9.620	1	2	115

Table 3 Critical shear stress for a stiffened panel ($\bar{\lambda}_{xx} = 0.78, \bar{\lambda}_{yy} = 0.23, \alpha_x = 2.25, \alpha_y = 7.30, c_x = -0.35, c_y = 0.014, \xi = 0.5, Z = 274, \nu = 0.33$

Range of m and n		Symmetric buckling			Antisymmetric buckling			Computer CPU time, mlsec
		$k_{s_{cr}}$	Dominant term of the eigenvector $A_{k\ell}$		$k_{s_{cr}}$	Dominant term of the eigenvector $A_{k\ell}$		
m	n		k	ℓ		k	ℓ	
1-9	1-9	230.2	3	7	236.0	1	6	870
1-5	1-11	230.9	2	6	239.1	2	7	500
1-6	1-15	230.6	3	7	237.0	1	4	920
2-5	2-12	240.0	2	6	247.0	3	6	380
1-5	4-10	231.4	1	5	240.0	3	6	240

characteristic equation (and thus the size of the determinant is reduced), the accuracy of the eigenvalue is not affected appreciably. (The difference is smaller than ½%.) This is reasonable because in a Galerkin or Rayleigh-Ritz procedure the effect of the buckling shape on the eigenvalue is small. This conjecture is verified through numerous applications, and it forms the basis of the proposed computation procedure.

In order to illustrate the advantages of and the logic behind the proposed computational procedure, consider the following hypothetical case. Suppose that in a given problem the geometry is such that the most influencing term in the eigenvector corresponds to $A_{k\ell}$, with $k=2$ and $\ell=22$. If we were to use matrices $[A]$ and $[B]$ which include all influencing terms and started from m and n equal to one ($m=1$-4 and $n=1$-28), the size of the two determinants (symmetric and antisymmetric) would be 56×56. On the other hand, if we had retained only those terms for which the relative amplitudes (as compared to $A_{2,22}$) the size of the required determinants would be reduced appreciably. In most cases, the size is 10×10 ($m=1$-4 and $n=20$-24), and the accuracy for k_s is not sacrificed appreciably. The question that has to be answered is how you find the most influencing term in the displacement vector and consequently the terms that must be retained in producing matrices $[A]$ and $[B]$ (required determinant). This can be done starting arbitrarily with any small range (say $m=3$-7 and $n=2$-6). By calculating the corresponding eigenvector, we know the relative magnitude of all of the sequential terms. If the first term has an amplitude greater than 10% of the most influencing term (for this range), then either the initial m or initial n (or both) should be lowered. Similarly, if the last term is greater than 10% of the most influencing one, then either the final m or final n (or both) should be increased. This way, the size of the originally small determinant is expanded to a still small, but acceptably accurate, determinant. Note that contraction also is possible by the same procedure. Thus, by employing this procedure, with the solution of a few reduced-order matrices, we can zero in to the final acceptable set: in this case, the $m=1$-4 and $n=20$-24 determinant. The computer time required for accurately predicting k_s by solving several times reduced-size matrices is much smaller than the time required to solve the large-order matrices. This conclusion is reinforced by the fact that, in most problems, the position of the most influencing term is never known a priori. This implies that, if we want to predict k_s accurately by solving a single determinant, its size must be extremely large. The computational procedure proposed to handle all problems containing shear is outlined below (fully automated):

1) Choose the range of m and n according to some a priori knowledge concerning the effects of different geometric parameters. This is facilitated by the parametric study reported herein.

2) Compute k_s, X_1, and X_2 for the preceding range of m and n by employing Eqs. (4) and (6).

3) Check the elements of the eigenvectors X_1 and X_2, and generate the new range of m and n so as to zero in to a final set that includes only elements that influence the buckling mode. (The 10% criterion is used here.)

4) Once this final range is located, compute k_s by Eq. (4). This value is taken as the accurate prediction for the critical condition (output).

The proposed computational procedure is employed in performing a number of parametric studies. The computations were made with the aid of the Georgia Tech high-speed digital computer UNIVAC 1108. A systematic program of study was undertaken to assess 1) the effectiveness of the procedure, and 2) the effect of a number of geometric parameters on the needed range of m and n to predict accurately critical conditions for stiffened or unstiffened configurations under destabilizing loads that include shear. These parameters are the panel aspect ratio ξ, the curvature parameter Z and the contribution to the extensional and flexural stiffness of panel made by the stiffeners (four parameters). A large range was chosen for these parameters which enhances the generality of the conclusions. Only some of the generated data are presented in tabular form here. The conclusions are based on all generated data. In Table 1 the value of $k_{s_{cr}}$, the most influencing terms, and the computer CPU time in milliseconds are given for a number of m and n ranges for an unstiffened, thin, circular panel. The geometric parameters are given on the table. (The symbols are the same as those in Ref. 5.) The reported computer time is the total time required for both symmetric and antisymmetric $k_{s_{cr}}$ calculations. Similar results are presented in Tables 2 and 3. Table 2 deals with the case of an unstiffened, thin, circular panel under the combined action of shear and uniaxial compression (k_x) (an example taken from Ref. 2). Table 3 deals with a stiffened, circular panel under shear only. This example is taken from Ref. 1. The results presented on these three tables clearly demonstrate the effectiveness of the proposed method.

Among the most important conclusions of the present investigation, one may list the following:

1) A fully automated computational procedure is developed which accurately predicts critical conditions for stiffened and unstiffened thin circular panels under the action of destabilizing loads, which include shear, with reasonable computational time.

2) For both stiffened and unstiffened panels, the effect of both the panel aspect ratio ξ and the curvature parameter Z on the needed size of matrices (determinant) for accurately predicting critical conditions that include shear is significant.

3) Conversely from the preceding conclusion, the effect of the amount and type of stiffening is rather insignificant. This conclusion is based on the large amount of generated data.

These last two conclusions are very important in the optimization of stiffened panels under destabilizing loads that include shear, because they provide a means for the investigator to generate a large amount of data in the design space with relatively small computer time (see Ref. 5).

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Nonlinear Vibrations of Beams Considering Shear Deformation and Rotary Inertia

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Introduction

LARGE-amplitude free vibrations of slender beams have been analyzed by use of continuum[1,2] and finite-element[3] methods. The purpose of the present note is to study the effect of shear deformation and rotatory inertia on the large-amplitude vibrations of beams. The method followed is based on an appropriate linearization of the nonlinear strain-displacement relations. The linearized stiffness and mass matrices are derived using standard principles.[4] The resulting linear algebraic eigenvalue problem is solved by employing an iterative technique[3] to obtain the nonlinear frequencies. In this paper, both simply supported and clamped beams are considered. The ratios of nonlinear frequency (ω_{NL}) to linear frequency (ω_L) for the fundamental mode are obtained for various values of slenderness ratios and central amplitude ratios.

Finite-Element Formulation

The nonlinear strain-displacement relations of a beam including shear deformation are given by[5]

$$\epsilon_x = \tfrac{1}{2}(\mathrm{d}w/\mathrm{d}x)^2, \quad \psi_x = -[(\mathrm{d}^2w/\mathrm{d}x^2) + (\mathrm{d}\gamma/\mathrm{d}x)] \quad \text{and}$$

$$\epsilon_{\kappa z} = -x^{\epsilon_{xz}} \tag{1}$$

where ϵ_x and ψ_x are the direct strain and curvature, respectively, ϵ_{xz} is the shear strain, w is the lateral displacement, γ is the shear rotation, and x is the axial coordinate.

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